4.2 Decidability Results for Termination

Dienstag, 17. November 2015 09:31

<u>Thm 4.2.1.</u> (Undecidability of Halting Problem for TRSs)

Let $\mathcal{R}$ be a TRS and t be a term.
The question whether t does not have infinite reductions w.r.t. $\to_{\mathcal{R}}$ is undecidable, but semi-decidable. The question whether $\mathcal{R}$ terminates (universal halting problem, i.e., whether all terms only have finite reductions) is not even semi-decidable.

<u>Proof</u>: Term rewriting is Turing-complete. Therefore, the undecidability results for Turing-complete languages also hold for term rewriting. $\boxtimes$

We start with a special case of TRSs where termination is decidable: right-ground TRSs, i.e., no variables on right-hand sides of rules.

<u>Ex. 4.2.2</u>

$$\begin{aligned}
\mathrm{and}(\mathrm{true},\mathrm{true}) &\to \mathrm{true}\\
\mathrm{and}(x,\mathrm{false}) &\to \mathrm{false}\\
\mathrm{and}(\mathrm{false},x) &\to \mathrm{and}(\mathrm{true},\mathrm{not}(\mathrm{true}))\\
\mathrm{not}(\mathrm{false}) &\to \mathrm{true}\\
\mathrm{not}(\mathrm{true}) &\to \mathrm{and}(\mathrm{false},\mathrm{false})
\end{aligned}$$

The TRS is not terminating:

$not(true) \to \underline{and(f,f)} \to and(t, n(t)) \to and(t, \underline{and(f,f)}) \to \ldots$

<u>Lemma 4.2.3</u> (Termination of right-ground TRSs)

Let $\mathcal{R}$ be a right-ground TRS. Then $\mathcal{R}$ terminates iff there is no rule $l \to r \in \mathcal{R}$ with $r \xrightarrow[\mathcal{R}]{+} t$ such that $t \trianglerighteq r$.

<u>Proof</u>: "$\Rightarrow$": If $r \xrightarrow[\mathcal{R}]{+} t$ with $t \trianglerighteq r$, then $\mathcal{R}$ does not terminate:

$$r \xrightarrow[\mathcal{R}]{+} t = t[r]_\pi \xrightarrow[\mathcal{R}]{+} t[t]_\pi = t[t[r]_\pi]_\pi$$
$$\xrightarrow[\mathcal{R}]{+} t[t[t]_\pi]_\pi \quad \text{etc.}$$

"$\Leftarrow$": Let there be no $l \to r \in \mathcal{R}$ such that $r \xrightarrow[\mathcal{R}]{+} t$ with $t \trianglerighteq r$. We have to show that $\mathcal{R}$ terminates by induction on the number of rules in $\mathcal{R}$.

<u>Ind. Base</u>: $\mathcal{R} = \emptyset \rightsquigarrow \mathcal{R}$ is trivially terminating.

<u>Ind. Step</u>: $\mathcal{R} \neq \emptyset$. Assume that $\mathcal{R}$ is not terminating. Let t be a minimal term that starts an infinite reduction (i.e., t does not terminate, but all proper subterms of t terminate).

$$t = t_0 \to_{\mathcal{R}} t_1 \to_{\mathcal{R}} t_2 \to_{\mathcal{R}} \ldots$$

Since t is minimal, there must be some reduction step at the root position:

$$t = t_0 \to_{\mathcal{R}} \ldots \to_{\mathcal{R}} l\sigma \to_{\mathcal{R}} r\sigma \to_{\mathcal{R}} \ldots \quad \text{with } l \to r \in \mathcal{R}$$

$s\sigma \underset{r}{\sim} s\sigma$

since r does not contain variables.

Thus: there is a rule $l \to r \in \mathcal{R}$ such that r starts an infinite reduction: $r \to_{\mathcal{R}} t_{i+1} \to_{\mathcal{R}} t_{i+2} \to_{\mathcal{R}} \dots$

<u>Case 1</u>: The rule $l \to r$ is not used in the infinite reduction of r. $\rightsquigarrow$ $\mathcal{R} \setminus \{l \to r\}$ is also non-terminating. Contradiction to the ind. hypothesis, since $\mathcal{R} \setminus \{l \to r\}$ has less rules than $\mathcal{R}$.

<u>Case 2</u>: $l \to r$ is used in the infinite reduction of r:

$$r \to^*_{\mathcal{R}} t_j[l\sigma']_\pi \to_{\mathcal{R}} t_j[\underbrace{r\sigma'}_{r}] \to_{\mathcal{R}} \dots$$

Thus: $r \to^+_{\mathcal{R}} t_j[r]_\pi$ which contradicts the prerequisites. $\boxtimes$

The alg. RIGHT_GROUND_TERMINATION uses Lemma 4.2.3. in order to decide termination for right-ground TRSs.

In contrast to full term rewriting, termination is decidable for right-ground TRSs because of 2 reasons:

1. We only have to check the right-hand sides of (finitely many) rules. If they terminate, then all terms terminate.

   For general TRSs, this is not true:

$\mathcal{R}: \; f(a,x) \to f(x,x)$ with $x \in \mathcal{V}$, $a \in \Sigma_0$.

The rhs $f(x,x)$ is in normal form, but we have the infinite reduction $f(a,a) \to_{\mathcal{R}} f(a,a) \to_{\mathcal{R}} \dots$

2. If a rhs r is non-terminating, then we can detect after finitely many steps, because $r \to_{\mathcal{R}}^{+} t \trianglerighteq r$.

For general TRSs, this is not true.

$\mathcal{R}: \; f(x) \to f(s(x))$

We have $f(s(x)) \to_{\mathcal{R}} f(s(s(x))) \to_{\mathcal{R}} \dots$

and $f(s(x))$ is not a subterm of $f(s^2(x)), f(s^3(x)), \dots$

<u>Ex 4.24</u> Illustrate RIGHT_GROUND_TERM:

$\mathrm{and}(t,t) \to t$

$\mathrm{and}(x,f) \to f$

$\mathrm{and}(f,x) \to a(t,n(t))$

$n(f) \to t$

$n(t) \to a(f,f)$

$T_1 = \{t\}$ $T_2 = \{f\}$ $T_3 = \{\underline{a(t,n(t))}\}$ $T_4 = \{t\}$ $T_5 = \{\underline{a(f,f)}\}$

$\Downarrow$

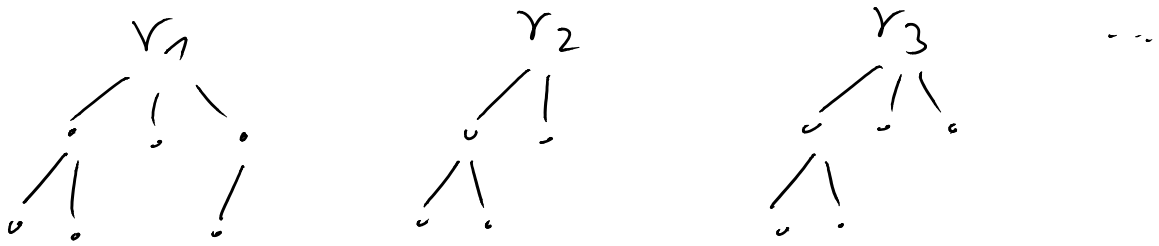
$T_1 = \emptyset$ $T_2 = \emptyset$ $T_3 = \{a(t, a(f,f))\}$ $T_4 = \emptyset$ $T_5 = \{f, a(t,n(t))\}$

$\Downarrow$

$T_1 = \emptyset$ $T_2 = \emptyset$ $T_3 = \{a(t,f), a(t, \underline{a(t,n(t))})\}$ $T_4 = \emptyset$ $T_5 = \{a(t, \underline{a(f,f)})\}$

<u>False</u>

So we simply construct search trees:



If TRS terminates: all these search trees are finite, alg stops after a while and returns "True"

If TRS doesn't terminate: by Lemma 4.2.3 some r_i reduces to a superterm of r_i $\rightsquigarrow$ alg. stops after a while and returns "False".

<u>Thm 4.2.5</u> (Decidability of Termination for Right-Ground TRSs)

Let $\mathcal{R}$ be a TRS with $\mathcal{V}(r) = \emptyset$ for all $l \to r \in \mathcal{R}$. Then termination of $\mathcal{R}$ is decidable and RIGHT_GROUND_TERMINATION is a decision procedure.